

Variable Force

Recall that:

Fact —

$$\text{Force} = \frac{d}{dt}(\text{momentum}) \quad (\text{N2})$$

$$\text{Impulse} = \Delta \text{momentum}$$

$$\text{Impulse} = \text{Force} \times \text{time}$$

Integrating both sides, we can obtain:

Fact (Impulse of a variable force) —

$$\text{Impulse} = \int \text{Force } dt = \Delta \text{momentum}$$

Example

A pebble of mass 0.2 kg is moving on a smooth horizontal sheet of ice. At time t seconds (where $t \geq 0$) a horizontal force of magnitude $2t^2$ N and constant direction acts on the pebble. When $t = 0$ the pebble is moving in the same direction as the force and has speed 6 ms^{-1} . When $t = T$ the pebble has speed 36 ms^{-1} .

Calculate the value of T .

$$\begin{aligned} & \Rightarrow \text{Impulse} = \Delta p \\ & \Rightarrow \int F dt = mV - mU \\ & \Rightarrow \int_0^T 2t^2 dt = 0.2 \cdot 36 - 0.2 \cdot 6 \\ & \Rightarrow \left[\frac{2}{3}t^3 \right]_0^T = 6 \\ & \Rightarrow T^3 = 9 \\ & \Rightarrow T = 2.080\dots \\ & \Rightarrow T = 2.08 \text{ (3 s.f.)} \end{aligned}$$

Example

A particle, A , of mass 500 g, is acted on by a variable force F N, which is defined as:

$$F = \begin{cases} 0.3t^2 + 0.5t & \text{for } 0 \leq t \leq 3 \text{ s} \\ t + 1.2 & \text{for } 3 < t \leq 5 \text{ s} \end{cases}$$

Find the speed of A after 4 seconds if the speed is 2 ms^{-1} when $t = 0$.

Vector Notation

Example

A toy sailing boat of mass 800 g is blown along by a constant wind, which produces a force of $(\mathbf{i} + 2\mathbf{j})\text{N}$ on the boat. If the boat is initially at rest, find its velocity after 10 s.

$$\text{Impulse} = \Delta \text{momentum}$$

$$\text{Force} \times \text{time} = \Delta \text{momentum}$$

$$\begin{aligned} \Rightarrow & 10 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = m\mathbf{v} - m\mathbf{u} \\ \Rightarrow & 0.8\mathbf{v} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \\ \Rightarrow & \mathbf{v} = \begin{pmatrix} 12.5 \\ 25 \end{pmatrix} = (12.5\mathbf{i} + 25\mathbf{j})\text{ms}^{-1} \end{aligned}$$

Example

A particle A of mass 2 kg is moving with velocity $(2\mathbf{i} + 3\mathbf{j})\text{ms}^{-1}$ when it collides with particle B, of mass 1.5 kg, which is moving with velocity $(\mathbf{i} + 2\mathbf{j})\text{ms}^{-1}$. If the particles coalesce during the collision:

- find the velocity of the combined particle after the collision
- find the total loss of energy as a result of the collision.

(a)

$$\begin{aligned} \text{COM :} & 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + 1.5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (2 + 1.5)\mathbf{v} \\ \Rightarrow & \begin{pmatrix} 5.5 \\ 9 \end{pmatrix} = 3.5\mathbf{v} \\ \Rightarrow & \mathbf{v} = \begin{pmatrix} 1.57 \\ 2.57 \end{pmatrix} \text{ms}^{-1} \text{ (3 s.f.)} \end{aligned}$$

(b)

$$\begin{aligned} \text{Initial Energy} &= \frac{1}{2}m_1|\mathbf{u}_1|^2 + \frac{1}{2}m_2|\mathbf{u}_2|^2 \\ &= \frac{1}{2} \cdot 2 \cdot \left| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right|^2 + \frac{1}{2} \cdot 1.5 \cdot \left| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right|^2 \\ &= (2^2 + 3^2) + \frac{3}{4}(1^2 + 2^2) \\ &= 16.75 \\ \text{Final Energy} &= \frac{1}{2} \cdot 3.5 \cdot \left| \begin{pmatrix} 1.57 \\ 2.57 \end{pmatrix} \right|^2 \end{aligned}$$

$$= \frac{7}{4} \left(\frac{11^2}{49} + \frac{18^2}{49} \right)$$
$$= \frac{445}{28}$$

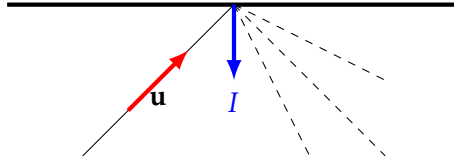
 \Rightarrow

$$\Delta Energy = -\frac{6}{7}$$

ie a loss of 0.857 J (3 s.f.)

Oblique Impacts

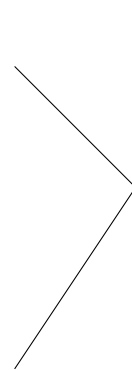
Consider an object colliding with a wall - where does it go?



Example

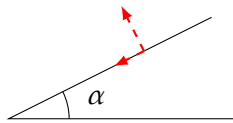
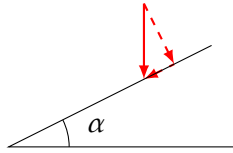
A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of the collision the direction of motion of S makes an angle of 60° with the wall. The coefficient of restitution between S and the wall is $\frac{1}{4}$. Find the speed of S immediately after the collision

$$\begin{aligned}
 N2(\downarrow) : & \quad u \cos 60^\circ = v \cos \alpha \\
 \Rightarrow & \quad v \cos \alpha = \frac{1}{2}u \\
 NEL(\leftarrow) : & \quad v \sin \alpha = eu \sin 60^\circ \\
 \Rightarrow & \quad v \sin \alpha = \frac{\sqrt{3}}{8}u \\
 \Rightarrow & \quad v^2 = u^2 \left(\frac{1}{4} + \frac{3}{64} \right) \\
 & \quad = \frac{19}{64}u^2 \\
 \Rightarrow & \quad v = \frac{\sqrt{19}}{8}u
 \end{aligned}$$



Example

A smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{1}{2}$. Immediately before striking the plane the ball has speed 5 ms^{-1} . The coefficient of restitution between the ball and the plane is $\frac{1}{2}$. Find the speed of the ball immediately after the impact.



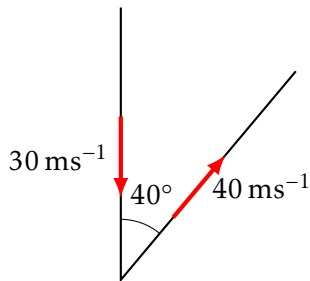
Before and after, the speed parallel to the slope is $5 \sin \alpha = \sqrt{5}$.

Perpendicular to the slope the final speed is $e 5 \cos \alpha = \frac{1}{2} \cdot 5 \cdot \frac{2}{\sqrt{5}} = \sqrt{5}$

Therefore the final speed is $\sqrt{10} \text{ ms}^{-1}$

Example

A cricket ball of mass 160 g is travelling horizontally at a speed of 30 ms^{-1} when it is hit by a cricket bat. It leaves the bat, horizontally, at a speed of 40 ms^{-1} travelling at 40° to the line of its original path, as shown in the diagram. What is the magnitude of the impulse of the bat on the ball?



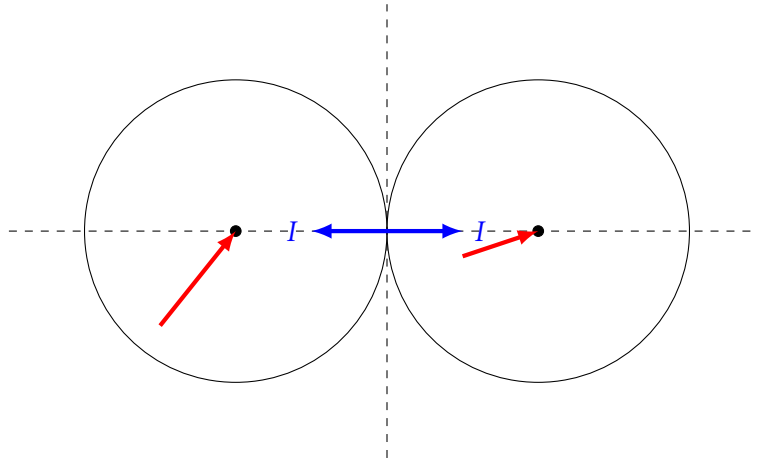
Method 1, we can use $\mathbf{I} = \Delta \mathbf{p}$ and use $\begin{pmatrix} 0 \\ -30 \end{pmatrix}$ and $\begin{pmatrix} 40 \cos 50^\circ \\ 40 \sin 50^\circ \end{pmatrix}$ as the velocities...

... we can draw the triangle from the vectors \mathbf{v} and $-\mathbf{u}$ and use the cosine rule

Oblique Collision of Two Spheres

Example

A sphere, A of mass 2 kg , is moving on a smooth horizontal surface with velocity $(4\mathbf{i} + 5\mathbf{j})\text{ms}^{-1}$ when it collides with a second sphere, B , of equal size and mass 3 kg , moving on the same surface with velocity $(3\mathbf{i} + \mathbf{j})\text{ms}^{-1}$. The line of the centres of the two spheres at the moment of collision is parallel to the \mathbf{i} -direction and the coefficient of restitution between the two spheres is 0.8 . Find the velocities of A and B immediately after the collision.



The vertical components of the velocities are left unchanged as the impulse acts along the line of centres (which is horizontal). Therefore we only need to consider the horizontal velocities.

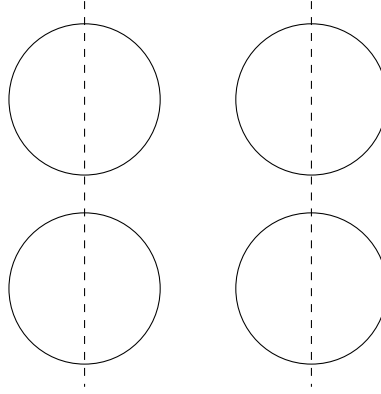
$$\begin{array}{ll}
 \text{NEL :} & v_B - v_A = 0.8(4 - 3) \\
 \text{COM :} & 2 \cdot 4 + 3 \cdot 3 = 2v_A + 3(v_A + 0.8) \\
 \Rightarrow & 5v_A = 17 - 2.4 = 14.6 \\
 \Rightarrow & v_A = 2.92 \\
 & v_B = 3.72
 \end{array}$$

Therefore A 's final velocity is $(\begin{smallmatrix} 2.92 \\ 5 \end{smallmatrix})\text{ms}^{-1}$ and B 's is $(\begin{smallmatrix} 3.72 \\ 1 \end{smallmatrix})\text{ms}^{-1}$

Example

A smooth sphere A of mass 5 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i} + 3\mathbf{j})\text{ m s}^{-1}$. Another smooth sphere B of mass 3 kg and the same radius as A is moving on the same surface with velocity $(4\mathbf{i} - 2\mathbf{j})\text{ m s}^{-1}$. The spheres collide when their line of centres is parallel to \mathbf{j} . The coefficient of restitution between the spheres is $\frac{3}{5}$.

Find the velocities of both spheres after the impact.



Considering the horizontal velocities for both spheres, there is no impulse in the \mathbf{i} direction so they are unchanged at 2 and 4 respectively.

$$\begin{array}{l}
 \text{NEL :} \\
 \text{COM}(\uparrow) : \\
 \Rightarrow \\
 \Rightarrow
 \end{array}
 \qquad
 \begin{array}{l}
 v_B - v_A = e \cdot 5 = 3 \\
 3 \cdot (-2) + 5 \cdot (3) = 3 \cdot (v_A + 3) + 5 \cdot v_A \\
 9 = 8v_A + 9 \\
 v_A = 0, v_B = 3
 \end{array}$$

Therefore $\mathbf{v}_A = 2\mathbf{i}$ and $\mathbf{v}_B = 4\mathbf{i} + 3\mathbf{j}$

Example

Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is $3m$ kg. The spheres are moving on a smooth horizontal plane and collide. Immediately before the collision the velocity of A is $(5\mathbf{j})\text{ms}^{-1}$ and the velocity of B is $(3\mathbf{i} - \mathbf{j})\text{ms}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i} + 2\mathbf{j})\text{ms}^{-1}$. Find

- (a) the speed of B immediately after the collision
 (b) a unit vector parallel to the line of centres of the spheres at the instance of the collision

(a)

$$\begin{aligned} \text{COM:} \quad & 2m \begin{pmatrix} 0 \\ 5 \end{pmatrix} + 3m \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 2m \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 3m\mathbf{v}_B \\ \Rightarrow & \begin{pmatrix} 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + 3\mathbf{v}_B \\ \Rightarrow & \mathbf{v}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Therefore the speed of B is $\sqrt{2}\text{ms}^{-1}$.

(b) In the collision A receives an impulse of

$$\begin{aligned} \text{Impulse} &= \Delta\mathbf{p} \\ \Rightarrow \text{Impulse} &= 2m \begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2m \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\ &= 2m \begin{pmatrix} 3 \\ -3 \end{pmatrix} \end{aligned}$$

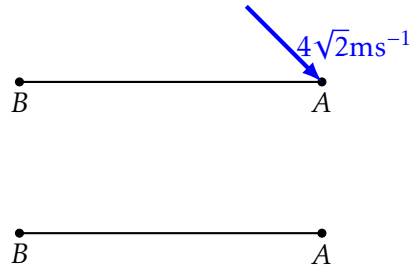
Since the impulse acts along the line of centres of the spheres, they must be along $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

Impulsive Tension in Strings

Example

In the diagram, the particles A and B , of masses 2.5 kg and 1.5 kg respectively, are joined by a light, inextensible string that is straight but not under tension. Particle B is initially at rest and particle A is projected at an angle of 45° to the string with speed $4\sqrt{2}\text{ ms}^{-1}$.

Find the speed of B immediately after the string goes taut. Find the impulse in the string.



First notice that since A and B are connected by the a taut string, they must be travelling at the same speed in the line of the string. Therefore before we have A travelling at $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$ and B at rest, and after we have B travelling at $\begin{pmatrix} v \\ 0 \end{pmatrix}$ and A travelling at $\begin{pmatrix} v \\ -4 \end{pmatrix}$.

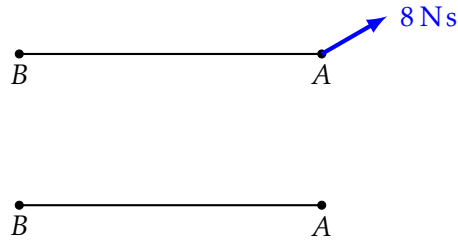
$$\begin{aligned} \text{COM}(\rightarrow): & & 0 \cdot 1.5 + 4 \cdot 2.5 &= 1.5v + 2.5v \\ \Rightarrow & & 4v &= 10 \\ \Rightarrow & & v &= 2.5\text{ ms}^{-1} \end{aligned}$$

$$\begin{aligned} \Rightarrow & & \text{Impulse} &= \Delta p \\ & & I &= 1.5 \cdot 2.5 - 0 \\ & & &= 3.75\text{ Ns} \end{aligned}$$

Example

A light, inextensible string has particles A and B , each of mass 400 g , attached to either end. The string and particles rest on a smooth table. The particle at A is hit with an impulse of 8 N s at an angle of 30° to the string, which is straight but not taut and at rest.

What is the velocity of B after the impulse at the instant the string becomes taut?



After the impulse, A has momentum $\begin{pmatrix} 8 \cos 30^\circ \\ 8 \sin 30^\circ \end{pmatrix}$ and velocity $\begin{pmatrix} 20 \cos 30^\circ \\ 20 \sin 30^\circ \end{pmatrix}$.

Immediately after the string goes taut, the horizontal speeds of both particles will be the same, ie

$$\begin{aligned} \text{COM}(\rightarrow) : & & 8 \cos 30^\circ &= 0.4v + 0.4v \\ \Rightarrow & & v &= 10 \cos 30^\circ \\ & & v &= 8.66 \text{ ms}^{-1} \end{aligned}$$

Example (OCR June 2019 - Q6)

Two particles A and B , of masses m kg and 1 kg respectively, are connected by a light inextensible string of length d m and placed at rest on a smooth horizontal plane a distance of $\frac{1}{2}d$ m apart. B is then projected horizontally with speed v m s⁻¹ in a direction perpendicular to AB .

- (a) Show that, at the instant that the string becomes taut, the magnitude of the instantaneous impulse in the string, I N s, is given by

$$I = \frac{\sqrt{3}mv}{2(1+m)}. \quad [5]$$

- (b) Find, in terms of m and v , the kinetic energy of B at the instant after the string becomes taut. Give your answer as a single algebraic fraction. [3]
- (c) In the case where m is very large, describe, with justification, the approximate motion of B after the string becomes taut. [2]

(a) Applying some geometric, we can see the string will go taught when B is at a point $(\frac{1/2d}{\sqrt{3}/2d})$ from A .

$$\begin{aligned} \text{COM}(\nearrow) : & \quad 0 + v \cos 30^\circ = (1+m)v_{\nearrow} \\ \Rightarrow & \quad v_{\nearrow} = \frac{\sqrt{3}v}{2(1+m)} \end{aligned}$$

And since $I = \Delta p$ and the mass of A is m we must have $I = \frac{\sqrt{3}mv}{2(1+m)}$.

(b)

$$\begin{aligned} k.e. &= \frac{1}{2} \cdot 1 \cdot ((v \sin 30^\circ)^2 + v_{\nearrow}^2) \\ &= \frac{1}{2} \left(\frac{v^2}{4} + \frac{3v^2}{4(1+m)^2} \right) \\ &= \frac{v^2((1+m)^2 + 3)}{8(1+m)^2} \\ &= \frac{(m^2 + 2m + 4)v^2}{8(1+m)^2} \end{aligned}$$

(c) In the basis of perpendicular to the string and parallel to the string the velocity is $(\frac{v/2}{\sqrt{3v}/(2(1+m))})$ and as $m \rightarrow \infty$ this becomes $\approx (\frac{v/2}{0})$, ie the particle starts to move at right angles to the string. Roughly we can think of this as a bit like moving in circular motion as the point is fixed and the particle is travelling at a fixed distance from it.